

APPROXIMATION BY EQUIVARIANT HOMEOMORPHISMS. I

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ABSTRACT. Locally linear (= locally smoothable) actions of finite groups on finite dimensional manifolds are considered in which two incident components of fixed point sets of subgroups either coincide or one has codimension at least three in the other. For these actions, an equivariant α -approximation theorem is proved using engulfing techniques. As corollaries are obtained equivariant “fibrations are bundles” and “controlled h -cobordism” theorems, as well as an equivariant version of Edwards’ cell-like mapping theorem and the vanishing of the set of transfer-invariant G -homotopy topological structures, rel boundary, on $T^n \times D_\rho$ (when T^n is the n -torus with trivial G action and D_ρ is a representation disc).

Here we consider locally linear (locally smooth **[Br]**) PL and topological actions of a finite group G on n -manifolds. In addition, we require that a G -manifold, M , have gaps of codimension ≥ 3 (i.e., for $H \subset G$, if M_α^H is a component of the H -fixed point set, M^H , and $M_\alpha^{>H} \subset M_\alpha^H$ is the subspace of points x for which the isotropy subgroup, G_x , strictly contains H , then either $M_\alpha^{>H} = M_\alpha^H$ or $M_\alpha^{>H}$ has codimension ≥ 3 in M_α^H). We shall use the term “not necessarily locally linear G -manifold” when we wish to drop the local linearity hypothesis but retain the property that if $M_\alpha^H \subset M_\beta^K$ but $M_\alpha^H \neq M_\beta^K$ then M_α^H is a locally flat (inequivariantly) submanifold of M_β^K of codimension at least three.

This, while self-contained, is the second in a series of papers **[SW_{2,...,s}, S]** in which we analyze the extent of failure of the topological invariance of equivariant Whitehead torsion and the consequent failure of subgroups of the equivariant Whitehead group $\text{Wh}_G^{PL}(M)$ of Illman **[II₁]** (cf. Rothenberg **[R]**) to classify PL or smooth G - h -cobordisms up to topological equivalence (under our gap hypotheses the arguments of Browder and Quinn **[BQ]** and Rothenberg **[R]** show that G - h -cobordisms are classified up to PL or smooth equivalence by such subgroups). We eventually conclude in **[SW₇, S]** (cf. **[SW₂]**) that the topological equivalence classes of G - h -cobordisms on a G -manifold M that are products over the union of fixed point components of M having dimension < 5 are classified by a group $\text{Wh}_G^{\text{Top}}(M)$ to which $\text{Wh}_G^{PL}(M)$ maps homomorphically. The image in $\text{Wh}_G^{\text{Top}}(M)$ of the pertinent subgroup of $\text{Wh}_G^{PL}(M)$ classifies up to equivariant homeomorphism, rel M , those G - h -cobordisms on M that admit equivariant handlebody decompositions relative to M . (This homomorphism is not generally surjective and there are G - h -cobordisms that do not admit such handle decompositions (cf. **[SW₅, S]**). In order

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to achieve our classification, we must develop a flexible tool for producing equivariant homeomorphisms which does not depend on explicit handle manipulations.

This is what we do in this paper: we develop the full equivariant analogue of the α -approximation theorem of Chapman and Ferry [CF] and draw the expected corollaries, including an equivariant topological “thin h -cobordism theorem” with control in the base of the cobordism. Our results are equivariant in the strong sense that nonisovariant input data is used to produce G -homeomorphisms. Thus, our results do not follow automatically from the inequivariant ones by induction on strata. Moreover, many of the inequivariant arguments do not generalize directly, as relevant equivariant obstruction groups are nonzero. For instance, as shown in [SW₂], even the topological Whitehead groups of the n -torus with the trivial action are nonzero (as are $\tilde{K}_{0G}(S^n)$ for all n , and even $\tilde{K}_{0G}^{\text{Top}}(S^1)$).

Our approach is to develop equivariant controlled engulfing techniques which enable us both to deduce an equivariant cell-like mapping theorem (Theorem 3 below) from the inequivariant theorem of Siebenmann [Si₁] and to deduce the equivariant α -approximation theorem from it by the Chapman “shuffle” as in [C₁]. In light of [E], in which Edwards gives a purely engulfing proof of the inequivariant cell-like mapping theorem, our results may be derived entirely through engulfing techniques.

We use “ G -equivalence” to mean an equivariant homotopy equivalence. For an open G -cover, α , of Y (i.e., $gU \in \alpha$ for all $U \in \alpha$) we say that a G -equivalence $f: X \rightarrow Y$ is a G - α -equivalence provided f is proper and there is an equivariant homotopy inverse for which the homotopies to the identity maps are limited by the covers α of Y and $f^{-1}\alpha$ of X .

THEOREM 1 (EQUIVARIANT α -APPROXIMATION). *Let N^n be a not necessarily locally linear G -manifold. Then for each open G -cover, α , of N there is an open G -cover, β , of N such that every G - β -equivalence, $f: (M^n, \partial M) \rightarrow (N, \partial N)$, from a (locally linear) G -manifold, M , of dimension n that restricts to a homeomorphism, $f: M_\gamma^N \rightarrow N_{f(\gamma)}^H$, on all fixed point components (of M or of ∂M) that have dimension < 5 is equivariantly α -homotopic to an equivariant homeomorphism.*

ADDENDUM (EQUIVARIANT β -DOMINATION). By further restricting β , we need assume only that f be a proper G - β -domination (cf. [F]), provided that $\dim M_\gamma^H = \dim N_{f(\gamma)}^H$ for each fixed-point component, M_γ^H , of M .

REMARK. The above theorem may be used to detect locally linear actions among the topological ones satisfying our standing dimension and codimension hypotheses.

Amusingly, we may now deduce a surgery-theoretic result similar to that used by Siebenmann in his proof of the cell-like mapping theorem. (See [KS, Essay V, Appendix B] for a complete bibliography.)

COROLLARY 1. *Let $f: M \rightarrow T^n \times D_\rho$ be a G -homotopy topological structure, $\text{rel } \partial$, on the product of the n -torus with the trivial action and a representation disc with codimension 3 gaps (i.e., M is a topological G -manifold and f is a G -equivalence which restricts to a homeomorphism on the boundary). If $\dim D_\rho^G + n \geq 5$, then there is a covering, $p: T^n \rightarrow T^n$ such that the transfer*

$$(p \times 1)^* f: (p \times 1)^*(M) \rightarrow T^n \times D_\rho$$

is G -homotopic $\text{rel } \partial$ to a G -homeomorphism.

PROOF. Choose a cover β of $T^n \times D_\rho$ such that any G - β -equivalence is G -homotopic to a homeomorphism and such that near $T^n \times 0$, β is of the form $\beta_1 \times \varepsilon$ for some ε , with β_1 a cover of T^n . Now choose p so that $(p \times 1)^*(M) \rightarrow T^n \times D_\rho \xrightarrow{\text{proj}} T^n$ is a G - β_1 -equivalence, and radially deform $(p \times 1)^*f$ to be a homeomorphism over $T^n \times (D_\rho - (\varepsilon/2)D_\rho)$.

We also get the expected equivariant analogue of the bundle theorem of [CF]. Here a G -fibration is a Hurewicz fibration in the category of G -spaces and a G -bundle is a bundle with fiberwise G -action which admits equivariant trivializations with the diagonal action over a cover by slices [LR]. We say that a G -space X is equivariantly locally path connected if for each $x \in B$ and each slice neighborhood U of x , there is a slice neighborhood $V \subset U$ of x such that for each $y \in V$ there is a path in U^{G_y} from y to x .

THEOREM 2 (EQUIVARIANT BUNDLE THEOREM). *Let $p: E \rightarrow B$ be a G -fibration where E and B are locally compact metric spaces and B is locally finite dimensional and equivariantly locally path connected. Suppose that each $p^{-1}b$ is a compact G_b -manifold of dimension n for which each fixed-point component has dimension ≥ 5 and that the restriction of p to $\bigcup_{b \in B} \partial p^{-1}b$ is a G -bundle. Then p is a G -bundle.*

We also obtain an equivariant version of Ferry's proof [F] of a conjecture of Kirby and Siebenmann. Here if β is a G -cover of X , a G - β -map $f: X \rightarrow Y$ is a G -map whose point inverses are limited by β .

COROLLARY 2. *Let α be an open G -cover of the G -manifold M^n . Then there is an open G -cover β of M such that if N^n is a G -manifold and $f: (M, \partial M) \rightarrow (N, \partial N)$ is a proper G - β -map which restricts to a homeomorphism on strata of dimension less than 5, then f is G -homotopic to a G -homeomorphism through α -maps.*

Let $f: X \rightarrow Y$ be a G -map between G -ANR's [J, L]. We say that f is G -cell-like (abbreviated G -CE) if for each $y \in Y$ and each G_y -neighborhood U of $f^{-1}y$, the inclusion $f^{-1}y \subset U$ is G_y -nullhomotopic. (Equivalently, for each orbit $G/H \subset Y$ and each G -neighborhood U of $f^{-1}(G/H)$, the inclusion $f^{-1}(G/H) \subset U$ G -deforms onto an orbit.) When X and Y are additionally separable and metric this is equivalent to f being an equivariant fine homotopy equivalence (i.e., a G - α -equivalence for all open G -covers α of Y) [SW₆], cf. [Hav].

We say that $f: X \rightarrow Y$ is an equivariant near homeomorphism if for each open G -cover α of Y there is a G -homeomorphism $h: X \rightarrow Y$ which is α -close to f . Our proof of Theorem 1 requires the following generalization of Siebenmann's cell-like mapping theorem [Si₁].

THEOREM 3 (EQUIVARIANT CELL-LIKE MAPPING THEOREM). *Let $f: (M, \partial M) \rightarrow (N, \partial N)$ be a G -CE map (as pairs) from a G -manifold to a not necessarily locally linear G -manifold which restricts to a homeomorphism on all fixed point components M_α^H of dimension less than 5. Then f is an equivariant near homeomorphism.*

COMPLEMENT. For f as above and α any open G -cover on N there is an equivariant α -homotopy $f_t: M \rightarrow N$ with $f_1 = f$ and f_t a G -homeomorphism for all $t < 1$.

In fact, we now get an equivariant version of Edwards CE mapping theorem [E] as an immediate corollary to [E] and Theorem 3. For simplicity, we state only the “manifold without boundary” case.

COROLLARY (EQUIVARIANT EDWARDS THEOREM). *Let X be a locally compact G -ANR such that for each unequal pair $X_\alpha^H \subset X_\beta^K$ of fixed point components, X_α^H has codimension ≥ 3 and is 1-LC in X_β^K . Then a G -CE map $f: M \rightarrow X$ of a G -manifold without boundary onto X that embeds $M^{<5}$ may be approximated $\text{rel } M^{<5}$ by equivariant homeomorphisms if and only if each fixed point component X_α^H of dimension ≥ 5 has the disjoint disks property.*

(Here, $M^{<5} = \{M_\alpha^H \mid \dim M_\alpha^H < 5\}$. A space Y has the disjoint discs property if each pair of maps $f, g: D^2 \rightarrow Y$ may be approximated by maps with disjoint images.)

The stratum by stratum induction in our argument leads us through an inequivariant extension of the α -approximation theorem to locally flat manifold k -ads, which may be of independent interest. A locally flat manifold k -ad, $(N; N_1, \dots, N_k)$ in a manifold N^n is a collection of submanifolds $N_i \subset N$ such that if $x \in \bigcap_{j=1}^p N_{i_j}$, there is a chart $\psi: \mathbf{R}^n \rightarrow N$ (or $\psi: \mathbf{R}_+^n \rightarrow N$) about $x = \psi(0)$ and a collection of linear subspaces, $V_{i_j} \subset \mathbf{R}^n$ such that $\psi|V_{i_j}$ (or $\psi|V_{i_j} \cap \mathbf{R}_+^n$) is a chart for x in N_{i_j} for $1 \leq j \leq p$. Thus, $(N; N_1, \dots, N_k, \partial N = N_{k+1})$ is a locally flat $(k+1)$ -ad, as is $(\partial N; \partial N \cap N_1, \dots, \partial N \cap N_k)$ a locally flat k -ad. We say that $(N; N_1, \dots, N_k)$ has codimension 3 gaps if each inclusion between components of intersections of N_i 's (including $N = N_0$) is either the identity or has codimension at least three. Maps and homotopies of k -ads are required to preserve the i th term for each i .

THEOREM 4 (α -APPROXIMATION FOR LOCALLY FLAT MANIFOLD k -ADS). *Let $(N^n; N_1, \dots, N_k)$ be a locally flat manifold k -ad with codimension 3 gaps. Then for each open cover, α , of N there is an open cover, β , of N such that each β -equivalence,*

$$f: (M^n; M_1, \dots, M_k, M_{k+1} = \partial M) \rightarrow (N; N_1, \dots, N_k, N_{k+1} = \partial N),$$

of $(k+1)$ -ads that restricts to a homeomorphism on all components of intersections of M_i 's with corresponding components in N of dimension less than 5 is α -close to a homeomorphism, h , of k -ads. Moreover, if f is a homeomorphism on $M_{i_1} \cap \dots \cap M_{i_p}$, h may be required to agree with it there.

REMARK. This is also true for locally flat G -manifold k -ads. Finally, our work in [SW_{2,4,5,7}] requires the following controlled h -cobordism theorem. Here, (W^{n+1}, M^n) is a G - h -cobordism if M and W are G -manifolds with $M \subset \partial W$ a codimension 0 submanifold with equivariantly bicollared boundary such that $M \rightarrow W$ and $\overline{\partial W - M} \rightarrow W$ are proper G -equivalences. We say that (W, M) is a G - α - h -cobordism, for α a G -cover of M , if there is an equivariant α -deformation retraction $r: W \rightarrow M$ such that the identity map of $\overline{\partial W - M}$ is equivariantly $(\beta, 1)$ -connected over M with control map r (cf. [Q₁]). (That is, r is a G -deformation retraction of W onto M whose tracks are limited by $r^{-1}\alpha$ and for each equivariant commutative

diagram

$$\begin{array}{ccc} S & \longrightarrow & \overline{\partial W - M} \\ \cap & & \downarrow \\ R & \xrightarrow{f} & M, \end{array}$$

where (R, S) is an equivariant relative 2-complex, there is an equivariant α -lift of f into $\overline{\partial W - M}$ rel S .) We say that W is a G - α -product (with respect to r) if there is a G -homeomorphism $h: (M \times I, M) \rightarrow (W, M)$ whose tracks $m \times I$ are limited by $r^{-1}\alpha$ (cf. $[\mathbf{Q}_1, \mathbf{C}_2]$).

THEOREM 5 (EQUIVARIANT CONTROLLED h -COBORDISM THEOREM). *Let α be an open G -cover of the G -manifold M . Then there is an open G -cover β of M such that if (W, M) is a G - β - h -cobordism which is (coherently) a G - β -product on all strata (of W) of dimension less than 6, then (W, M) is a G - α -product.*

One of our main tools is an equivariant version of the deformation principle of Edwards and Kirby $[\mathbf{EK}]$, as generalized by Siebenmann $[\mathbf{Si}_2]$. We discuss this and its corollaries in §1.

We give our controlled (isovariant) engulfing theorems in §2 and apply them to obtain a main engulfing lemma in §3. Theorems 3, 1, 2, and 5 are proven in §§4, 5, 7, and 8, respectively, with Corollary 2 and the addendum to Theorem 1 given in §6.

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The results of this paper are true in greater generality. With adjustments of the hypotheses to fit the new categories (e.g., proper or isovariant homotopy conditions), they hold for (1) a certain class of locally universal G -actions on Hilbert cube manifolds $[\mathbf{SW}_6]$ and (2) for proper Lie group actions on n -manifolds either with or without the codimensions hypothesis $[\mathbf{SW}_8]$.

1. Deformation of embeddings. In this section and the next, our G -manifolds need not satisfy gap hypotheses. All function spaces are assumed equipped with the compact-open topology. Given G -spaces X and Y , we consider the conjugation action of G on the space, Y^X , of all maps from X to Y , i.e., $(g^*f)(x) = gfg^{-1}(x)$, for $g \in G$, $f \in Y^X$, and $x \in X$. Thus, the fixed point set is the subspace of equivariant maps. Note that composition, $Z^Y \times Y^X \rightarrow Z^X$, is equivariant, and that the space, $\mathcal{H}(X)$, of homeomorphisms of X is a group in the category of G -spaces whenever composition and inversion are continuous.

We are concerned with equivariant deformations of G -subspaces of embeddings of (locally linear) G -manifolds. In this context, the work of Edwards and Kirby $[\mathbf{EK}]$ generalizes verbatim, as noted by Siebenmann $[\mathbf{Si}_2]$. We find Siebenmann's setting of it in terms of stratified spaces convenient, but, not being concerned with actions of connected groups and not always assuming isovariant data, we eventually find it simplest for our purposes to work in the manifolds, rather than the orbit spaces. Thus, we review parts of $[\mathbf{Si}_2]$ in this context.

A G -stratified set is a metrizable G -space, X , equipped with a filtration by closed G -subspaces, $\emptyset = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n)} \subset \cdots \subset X$, such that for each $n \geq 0$, the components of $X^{(n)} - X^{(n-1)}$ (which may be void) are open in $X^{(n)}$. The depth of the stratification is the length, if finite, of a maximal chain of components of pure strata, $X^{(n)} - X^{(n-1)}$, whose closures intersect.

Such a space, X , is an equivariantly conelike stratified set (G -CS) provided that

(i) each component of $X^{(n)} - X^{(n-1)}$ is inequivariantly an n -manifold without boundary and has constant isotropy subgroup, and

(ii) each $x \in X^{(n)} - X^{(n-1)}$ has an open slice neighborhood in X that is isomorphic as a G_x -stratified set to the product of a slice neighborhood of x in $X^{(n)} - X^{(n-1)}$ with the open cone, cL , on a compact G_x -stratified set, L , of finite formal dimension (i.e., with a finite number of strata). More general definitions are obviously possible, but this one is convenient and provides for simple proofs.

EXAMPLES. (i) Let M be a G -manifold, then M is a G -CS set with $M^{(n)} - M^{(n-1)}$ the union of all components of all $M^H - M^{>H}$ and $\partial M^H - (\partial M)^{>H}$ which have dimension n . Moreover, if X is any union of G -translates of components of M^H 's and $(M^H \cap \partial M)$'s, then X is a sub- G -CS set of M , as is ∂M . This is the *orbit type stratification* of M .

(ii) Let $N \subset M$ be a G -submanifold which is equivariantly locally flat in the sense that each point, x , in N has a slice in M that is G_x -homeomorphic to the product of a slice in N with a G_x -representation. Then there is a G -CS structure on M obtained by filtering both N and $M - N$ as above. This generalizes in the obvious way to locally flat G -manifold k -ads.

(iii) Each locally compact G -simplicial complex is a G -CS set with the filtration given by its skeleta.

We now proceed with [EK, Si₂]. For $C \subset U \subset X$, all G -spaces, let $\mathcal{E}(U, C; X)$ be the space of open embeddings of U in X which restrict to the inclusion on C .

PROPOSITION 1.1. *Let U be an open G -neighborhood of a compact G -subspace, C , of a G -CS set, X . Let A' be a closed G -neighborhood of the closed G -subspace, A , of X . Then there is a G -neighborhood, W , of the inclusion, $i: U \rightarrow X$, in $\mathcal{E}(U, U \cap A'; X)$ and a G -deformation,*

$$h_t: W \rightarrow \mathcal{E}(U, U \cap A; X),$$

of W into $\mathcal{E}(U, U \cap (A \cup C); X)$ such that

(i) *for $f \in W$ and x outside any preassigned compact G -neighborhood of C in U , $h_t(f)(x) = f(x)$ for all t , and*

(ii) *for all sub- G -CS sets, Y , of X and all $f \in W$ preserving or fixing Y , each $h_t(f)$ also preserves or fixes Y .*

SKETCH OF PROOF. As in [Si₂], it is enough to show that the proposition holds for U an element of a G -basis for X , and the same pair of handle lemmas will suffice, with $X = G \times_H (R^m \times cL)$, $U = G \times_H (10\dot{B}^m \times c_{10}L)$, and H acting trivially on R^m . The remarks in the second paragraph of the section indicate that Siebenmann's argument for $X = H \times_H (R^m \times cL)$ is H -equivariant, and the rest follows by translation.

Recall that neighborhoods of a map, $f: X \rightarrow Y$, defined by majorant functions, $\varepsilon: Y \rightarrow (0, \infty)$, have a basis given by the subsets, $N_\alpha(f) \subset Y^X$, of maps α -close to f as α ranges over the open covers of Y . If f is equivariant and α is a G -cover of Y , then $N_\alpha(f)$ is an invariant subset of Y^X .

COROLLARY 1.2. *Let X, U, A, A' , be as above with $i: U \subset X$, but let C be closed and not necessarily compact. Let α be an open G -cover of X , and let D be a closed neighborhood of C in U . Then there are an open G -cover, β , of X and*

a G -deformation of $N_\beta(i) \cap \mathcal{E}(U, U \cap A'; X)$ into $\mathcal{E}(U, U \cap (A \cup C); X)$, with the deformation taking place in $N_\alpha(i) \cap \mathcal{E}(U, U \cap A; X)$ and satisfying (ii) above as well as

(i') $h_t(f)(x) = f(x)$ for all t and for all $x \in X - D$.

COROLLARY 1.3. *The homeomorphism group, $\mathcal{H}(X)$, of a compact G -CS set, X , is locally G -contractible. If X is a noncompact G -CS set, then for each G -cover, α , of X , there is a G -cover, β , of X such that in $\mathcal{H}(X)$, $N_\beta(e)$ is null-homotopic in $N_\alpha(e)$, e being the identity.*

The following is a simple generalization of [Si₂, Theorem 6.5]. In it, U and X are G -CS sets, and $C \subset U$ is a closed, invariant set with compact frontier. $\mathcal{E}_C(U; X)$ denotes the space of open embeddings, $f: U \rightarrow X$, for which $f(C)$ is closed, $\mathcal{E}(C; X)$ is the space of closed embeddings of C in X , and $p: \mathcal{E}_C(U; X) \rightarrow \mathcal{E}(C; X)$ is restriction. If B is a G -space, we say that it is equivariantly (G -) path-connected if B^H is path-connected for each subgroup, H , of G . Given an equivariant map, $j: B \rightarrow \mathcal{E}_C(U; X)$, let $f_b = pj(b)$, and define $\pi_b: \mathcal{H}(X) \rightarrow \mathcal{E}(C; X)$ by $\pi_b(h) = hf_b$.

COROLLARY 1.4 (ISOTOPY EXTENSION THEOREM). *Let B be an equivariantly path-connected G -space, and let $j: B \rightarrow \mathcal{E}_C(U; X)$ be an equivariant map. If either X has finitely many components or B is G -locally connected, then the pull-back $(pj)^*\pi_b$ is a G -locally trivial principal bundle with fiber $\mathcal{H}(X, \text{rel } f_b(C))$, for each $b \in B^G$. Moreover, the same conclusion holds for the subspaces of these mapping spaces comprised of those maps which preserve or fix any family of sub- G -CS sets of X .*

PROOF. Fix $\phi \in B^G$. For arbitrary $b \in B$, the argument in [Si₂, Theorem 6.5 (I)] gives a G_b -equivariant section, s_b , of $(pj)^*\pi_b$, defined on a slice neighborhood, N , of b . Thus, for $x \in N$, $s_b(x) \in \mathcal{H}(X)$, and $s_b(x)f_b = f_x$. To get a local section of π_ϕ on N , we require a homeomorphism, $F_b: X \rightarrow X$, with $F_b f_\phi = f_b$. This will follow from the argument for the usual isotopy extension theorem [Si₂, Theorem 6.5 (II)] applied to a G_b -equivariant path from ϕ to b .

COROLLARY 1.5 (ISOTOPY FACTORIZATION THEOREM). *Let α be an open G -cover of the G -CS set X . Let $f: D_\rho \rightarrow \mathcal{H}(X)$ be equivariant and uniformly compactly supported, where D_ρ is the unit disc of a linear representation \mathbf{R}_ρ^n of G over \mathbf{R} . Then there are G -maps $f_1, \dots, f_k: D_\rho \rightarrow \mathcal{H}(X)$, with each f_i supported on a compact subset of some GU_i with $U_i \in \alpha$, such that $f(x)f(0)^{-1} = f_1(x) \circ \dots \circ f_k(x)$ for all $x \in D_\rho$.*

PROOF. Use an argument as given in [EK] for the zero dimensional representation and extend it by a Lebesgue number argument as in [Si₂, Theorem 6.5 (II)], employing Corollaries 1.2–1.4 as needed.

REMARKS. In the above, we may replace D_ρ by the universal G -Hilbert cube, Q_G (cf. [SW₆]), and hence by any compact G -AR.

As in [Si₂, Theorem 6.17] there is an equivariant extension of locally flat isotopies theorem. However, local flatness is difficult to verify unless the isotopy is supported on a stratum of the sub- G -CS set $Y \subset X$, in which case Corollary 1.5 yields a more finely controlled conclusion.

COROLLARY 1.6. *Let Y be a sub- G -CS set of X and let $f: D_\rho \rightarrow \mathcal{H}(Y)$ be an equivariant isotopy with support on a compact subspace K of a pure stratum of Y , with $f_0 = \text{id}$. Then f extends to an equivariant isotopy $F: D_\rho \rightarrow \mathcal{H}(X)$, $F_0 = \text{id}$, with support on any preassigned compact neighborhood of K . Moreover, if $\varepsilon > 0$ is given, we may assume that each $F(D_\rho)(x)$ is either constant or is contained in the ε -neighborhood of some $F(D_\rho)(y)$. Moreover, for each $\delta > 0$ there is a neighborhood U_δ of Y such that for each $x \in U_\delta$, there is a $y \in Y$ such that $F(-)(x)$ is δ -close to $f(-)(y)$ as functions $D_\rho \rightarrow X$.*

PROOF. Apply Corollary 1.5 to f with respect to a small cover of K by charts. Extend each of the resulting factors by an Alexander trick, damping them out along the cone coordinate on the cone on the link, cL , in the given chart, to be the constant isotopy outside of the γ -neighborhood of the cone point, where γ depends on the number of factors.

COROLLARY 1.7. *The same conclusion holds, with ε and δ replaced by an open G -cover α and β of X and with F supported on any closed neighborhood of Y , if f is a finite composite of disjoint unions of isotopies supported on compact subsets of pure strata of Y .*

2. Controlled equivariant engulfing. Once again, our G -manifolds need not satisfy gap hypotheses. Fix a G -manifold of dimension n and a proper, equivariant control map, $p: M \rightarrow B$, where B is a locally compact finite-dimensional metrizable space. In this section, we prove an isovariant engulfing theorem which combines the conclusions of [Il₂, Bi, and SGH].

Let P be a G -PL space and let $f: P \rightarrow M$ be a proper, isovariant G -map. We say that f is locally polyhedral if for each $x \in M$ there is a G_x -chart, $\alpha: \mathbf{R}_\rho^n \rightarrow M$, of M around x such that on $f^{-1}(\alpha(\mathbf{R}_\rho^n))$, the composite of f and α^{-1} is G_x -equivariantly PL.

The following trivially generalizes a lemma in [SGH].

LEMMA 2.1. *The image, $f(R)$, of a locally polyhedral map, $f: R \rightarrow M$, inherits unique equivariant PL structure such that $f: R \rightarrow f(R)$ is PL and $f(R) \subset M$ is locally polyhedral.*

Let $f: K \rightarrow M$ be a proper, isovariant map of a G -CW complex into M . We say f has coherent codimension $\geq k$ if $\dim M_{f(\gamma)}^H - \dim K_\gamma^H \geq k$ for each component K_γ^H of K^H .

Let $p \leq n - 3$, and let \mathcal{F} be a family of orbit types occurring in M . An engulfing structure, (M, U) , on M , of type (β, p, \mathcal{F}) consists of collections $U_0 \subset \cdots \subset U_p$ and $M_0 \subset \cdots \subset M_p$ of open subspaces of M , with $U_i \subset M_i$, such that if $f: (K, L) \rightarrow (M_i, U_i)$ is a proper (in M) isovariant map of coherent codimension ≥ 3 , with (K, L) a G -CW pair of relative dimension $\leq p - i$ and $K - L$ of type \mathcal{F} , then there is an isovariant $p^{-1}\beta$ -homotopy, f_t , of f , rel L in M_{i+1} to a map into U_{i+1} .

We first give a controlled version of the inequivariant engulfing theorem of [SGH], and then give some equivariant versions. (A full generalization will require an equivariant version of the controlled taming theory of Miller [Mil], Bryant [Bry], and Bryant and Seebeck [BS], and its applicability will depend on the existence of equivariant topological dual skeleta, due inequivariantly to Edwards (see [SGH]).)

The PL requirement in the last sentence of the following is due to the lack of an ε -taming theory for half-spaces.

PROPOSITION 2.2 (CONTROLLED TOPOLOGICAL ENGULFING). *Let $p \leq n - 3$ and let α be an open cover of the control space B . Then there is an open cover β of B such that for each engulfing structure (M, U) on M of type (β, p) and each locally polyhedrally embedded pair $(P, Q) < (M_0, U_0)$ for which $\dim P \leq n - 3$, $\dim P - Q \leq p$, $P \cap \partial M \subset Q$ and $\overline{P - Q} \subset M$ is a proper map, there is a $p^{-1}\alpha$ -isotopy, h_t , rel Q , of the identity map of M , supported on a closed (in M) subspace of M_p , with $P \subset h_1(U_p)$. When M is PL and $P \subset M$ is PL, the requirement that $P \cap \partial M \subset Q$ may be replaced by the requirements that $n > 5$, $\dim(P \cap \partial M) < n - 3$, and that the homotopies f_t given by the engulfing structure may be chosen to preserve the boundary.*

PROOF. For M topological, we may assume by passage to $(M - \partial M, U - \partial M)$ that $\partial M = \emptyset$. For simplicity, we may assume that α is relatively compact, so that any $p^{-1}\alpha$ -homotopy of a proper map is proper. We argue by induction on p . For $p < n - 3$, the result is almost immediate, using the global controlled taming theory of [SGH] to get the general position data to apply the proof of Bing's engulfing Theorem A [Bi] (details below). Here, β may be any p -fold star refinement of α . Briefly, let $R = \overline{P - Q}$, let $S = P \times 0 \cup R \times I$ and let $h: S \rightarrow M$, be induced by the homotopy f_t of the engulfing structure. Let H be a $p^{-1}\beta$ -homotopy approximating h rel $P \times 0$ which is locally polyhedral and in (Hudson-type [H]) general position as in [SGH] and let L be the union of the vertical line segments through points of the singular set of H . Since $\dim L < p$, induction gives an isotopy k_t limited by the $(p - 1)$ -fold star, $\text{St}^{p-1}(p^{-1}\beta)$, of $p^{-1}\beta$, with $L \subset k_1(U_p)$. Removing the interior of a regular neighborhood of L in $(k_1H)^{-1}(U_p)$, we may assume H is an embedding, and engulf up the vertical skeleta of a cylindrical (projection onto $P \times 0$ is simplicial) triangulation T of S whose simplices have small enough diameter that the resulting isotopy is limited by $p^{-1}\beta$.

For $p = n - 3$, the proof is modeled on Bing's engulfing Theorem D [Bi]. Here, β is a $(p + d + 1)$ -fold star refinement of α , where $d = \dim B$, with the additional property that $\beta = \beta_1 \cup \cdots \cup \beta_{d+1}$ where each β_i is a discrete collection of subsets of B . Here, $\dim S = n - 2$, so we may not be able to obtain a locally polyhedral approximation to h as above. We do obtain a $p^{-1}\beta$ -homotopy, H , approximating h , with the following properties with respect to a fine cylindrical triangulation T of S as above.

(1) H is locally polyhedral and in general position on the $(n - 3)$ -skeleton, T^{n-3} , of T .

(2) For each $(n - 2)$ -simplex, σ , of T there is a chart for which $H|_{\sigma \cup T^{n-3}}$ is PL, and $\sigma - \partial\sigma$ is in general position with respect to T^{n-3} .

(3) H is simplicially nondegenerate on T (i.e., H embeds each simplex of T).

(4) There is a shadow $L \subset S$ for H , i.e., if X_1 is the closure of $\{x \in S \mid H(x) = H(y) \text{ for some } y \in T^{n-3}\}$ then L is a $(p - 1)$ -dimensional union of vertical segments, containing X_1 , such that if $x, y \in S$ with $H(x) = H(y)$, then $x \in L$ if and only if $y \in L$.

This data is sufficient for the engulfing. Induction gives a $\text{St}^{p-1}(p^{-1}\beta)$ isotopy engulfing $((P^{p-2} \times I) \cap S) \cup L$, and the argument above (modeled on Bing's engulfing

Theorem B) gives a further $p^{-1}\beta$ isotopy engulfing $(P^{p-1} \times I) \cap S$. The (embedded) $(n-2)$ -simplices of T are now engulfed vertically in tracks along sets of β_1, β_2 , etc., giving a $\text{St}^d(p^{-1}\beta)$ -isotopy engulfing T^{n-3} .

We first show how to obtain (1)–(3). Start with a locally finite cover $\mathcal{V} = \{V_i\}_{i \geq 0}$ of M by charts. Choose a closure refinement of the cover $h^{-1}\mathcal{V}$ of S , and, via simplicial neighborhoods, choose a cover $\mathcal{C} = \{C_i\}$ of S by codimension zero PL subspaces with bicollared frontiers, ∂C_i (which thus have dimension $\leq n-3$) such that $h(C_i) \subset V_i$ and $C_i \cap C_j = \partial C_i \cap \partial C_j$. Now choose a fine triangulation, T_1 , of S , for which each C_i is a subspace and choose an approximation, h_1 , of h , satisfying (1), such that $h_1: C_i \rightarrow V_i$ is PL (after a possible change in the PL structure of V_i) and in general position. By Rushing's relative codimension 3 ε -taming theorem [Ru, Theorem (5.5.2)] there is a small homeomorphism $g: V_i \rightarrow V_i$, $\text{rel } C_i \cap T_1^{n-3}$, such that $g^{-1}(V_i \cap T_1^{n-3})$ is PL. Using g both to alter the PL structure of V_i and to shift h_1 on the interior of the $(n-2)$ -simplices of C_i , a small shift on the $(n-2)$ -simplices, $\text{rel } \partial$, now achieves (1) and (2) for an approximation h_2 . Now choose a cylindrical subdivision, T , of T_1 for which each $h_2: C_i \rightarrow V_i$ is simplicially nondegenerate (guaranteed by general position) and repeat the taming moves above inductively over the $(n-2)$ -simplices of T_1 to regain (1) and (2) for the new triangulation, obtaining an approximation, h_3 , satisfying (1)–(3).

Let $k = h_3$. By (1) and (2), X_1 is a subpolyhedron of S of dimension $\leq n-5$, with $k \mid X_1$ PL onto $k(T^{n-3})$. Let L_1 be the union of the vertical segments through X_1 . Make another set of taming moves relative to T^{n-3} so that each $(n-2)$ -simplex is PL with respect to L_1 and shift each $(n-2)$ -simplex $\sigma \text{ rel } k(T^{n-3})$ so that $\sigma - (\partial\sigma \cup (\sigma \cap X_1))$ is in general position with respect to L_1 . Call the new map k_1 and note that we have not changed the intersections involving T^{n-3} and that k_1 satisfies (1)–(3) with T^{n-3} replaced by $T^{n-3} \cup L_1$ in (2). Let X_2 be the singularities of k_1 involving $T^{n-3} \cup L_1$ and let L_2 be the union of the vertical segments through X_2 . Since $\dim L_1 \leq n-4$, $\dim(X_2 - X_1) \leq n-6$. Repeat the process inductively until $X_{j+1} = X_j$. The map k_j will satisfy (1)–(4), with shadow L_j .

COROLLARY 2.3 (CONTROLLED EQUIVARIANT ENGULFING). *Let $p \leq n-3$ and let α be an open G -cover of the control space B . Then there is an open G -cover, β , of B , such that for each engulfing structure (M, U) on M of type (β, p, \mathcal{F}) , the following conditions on an equivariantly locally polyhedrally embedded pair $(P, Q) \subset (M_0, U_0)$ provide for the existence of a $p^{-1}(\alpha)$ - G -isotopy, h_t , $\text{rel } Q$, of the identity map of M , supported on a closed (in M) subspace of M_p , for which $p \subset h_1(U_p)$.*

(1) $P \subset M$ has coherent codimension ≥ 3 , $P - Q$ is of type \mathcal{F} and of dimension $\leq p$, $\overline{P - Q} \subset M$ is a proper map, and for any stratum, N , of M or ∂M of dimension less than 5, $P \cap N \subset Q$.

(2) If M is topological or if $P \subset M$ is not PL, then each component of $P - Q$ has only one orbit type and $P \cap \partial M \subset Q$. If M and $P \subset M$ are PL and $P \cap \partial M \not\subset Q$ then $P \cap \partial M \subset \partial M$ satisfies (1), and the homotopies obtained from the engulfing structure may be chosen to respect the boundary.

PROOF. When M and $P \subset M$ are PL, the general positioning for the argument above is given by Illman [Il3]. In the general case, we may consider each stratum of $P - Q$ disjointly from the others, and by passage to the strata of (M, U) , may

assume that P itself has a single orbit type and simply extend equivariantly the moves given by passage to the orbit space.

REMARKS. More general equivariant topological engulfing is at present useless, without the existence of equivariant topological dual skeleta. However, in case of need, we may obtain more general engulfing by a sequence of engulfing structures $(M^1, U^1) \subset \cdots \subset (M^k, U^k)$, where k is the depth of the stratification (by dimension) of M by orbit types, (M^i, U^i) is an engulfing structure of type $(\beta, p_i, \mathcal{F})$, where \mathcal{F} is the family of fixed point components of a given dimension, and $(M_{p_i}^i, U_{p_i}^i) \subset (M_0^{i+1}, U_0^{i+1})$.

3. The main lemma. Our main applications of controlled engulfing are given by the following lemma. Here, we consider G -ANR's X . (An action of a finite group G on a finite-dimensional space X is a G -ANR if X and each X^H are ANR's.) We denote by X^* some closed, G -ANR subset of X containing $\bigcup_{H \neq 1} X^H$ that is 1- LC embedded in X , i.e., for all $\varepsilon > 0$ there is $\delta > 0$ such that maps $f: S^i \rightarrow X - X^*$ of diameter $< \delta$ extend to maps of D^{i+1} of diameter $< \varepsilon$.

LEMMA 3.1. *Let α be an open G -cover of the G -ANR N and let $V \subset N$ be a G -neighborhood of N^* . Then there is an open G -cover β of N and a G -neighborhood $V' \subset V$ of N^* such that if M^n is a G -manifold and $f: M^n \rightarrow N^n$ is a G - β -equivalence which restricts to a homeomorphism $f: M^* \rightarrow N^*$, and U is a G -neighborhood of M^* , then there is an equivariant $f^{-1}\alpha$ -isotopy, h_t , of the identity map of M , supported on a closed (in M) subset of $f^{-1}V - M^*$, with $f^{-1}(\bar{V}') \subset h_1(U)$.*

REMARKS. In the special case where f is the identity map of N , the statement says precisely that the neighborhoods of N^* in N satisfy a controlled equivariant version of the I -axiom of [SGH]. Thus, I -regular neighborhoods may be used (with control) to study situations where Quinn's mapping cylinder neighborhood obstruction is nontrivial, which may be the case even for G -manifolds (e.g. [Q₂, 2.1.4]).

PROOF OF LEMMA 3.1. For simplicity of notation we shall assume unless stated otherwise that all inclusions of neighborhoods, $V \subset W$, have the property that $\bar{V} \subset W$.

We construct a sequence of neighborhoods of N^* , $V' = V_{0,n-2} \subset \cdots V_{0,0} \subset V_{1,0} \subset \cdots \subset V_{1,2} = V$ on which the size of β will depend, such that for f and U as above,

(a) There is a sequence, $W_0 \subset \cdots \subset W_2 = U \cap f^{-1}V'$ of neighborhoods of M^* such that taking $M_i = f^{-1}V_{1,i} - M^*$ and $U_i = W_i - M^*$, $i = 0, 1, 2$, gives an engulfing structure of type $(\eta, 2, \{e\})$, where the size of η is determined by Corollary 2.3 and the argument below.

(b) Taking $M_i = f^{-1}V_{0,0} - M^*$ and $U_i = f^{-1}(V_{0,0} - \bar{V}_{0,i+1})$ for $i = 0, \dots, n-3$ gives an engulfing structure of type $(\eta, n-3, \{e\})$.

To complete the proof, let $\varepsilon: M \rightarrow [0, \infty)$ be equivariant, with $\varepsilon^{-1}(0) = M^*$, so that, on $M = M^*$, the cover by ε -balls refines both $f^{-1}\beta$ and the distance from M^* . Choose equivariant topological dual skeleta (by pullback from the orbit space via [SGH]) S^2 and T^{n-3} for $M - M^*$ with respect to ε .

Apply Corollary 2.3 via (a), with P = the largest subcomplex of S contained in $f^{-1}(V_{1,0})$ and Q = the largest subcomplex of S contained in W_0 , obtaining an

isotopy h_t . Let $W_{-1} \subset W_0$ be a neighborhood of M^* on which h_t is the identity and let $W_{-2} \subset W_{-1}$ be such that the ε -neighborhood of $M - W_{-1}$ is contained in $M - W_{-2}$. The argument now proceeds as in [SGH, Theorem 2.1], with $Z = f^{-1}\overline{V}_{0,0} - W_{-2}$ and \hat{Z} equal to the closed ε -neighborhood of Z : there is a small isotopy, g_t , of 1_M , supported on a closed subspace of $f^{-1}V_{0,0} - M^*$, with $\hat{Z} \cap T \subset g_1(M - f^{-1}(\overline{V}'))$. Successive applications of h_t , an ε -push, and g_t^{-1} engulfs $f^{-1}(\overline{V}')$ by W_2 .

We first construct $V_{1,0} \subset \cdots \subset V_{1,2}$. Choose β_1 small with respect to η and choose neighborhoods $N_0 \subset \cdots \subset N_{11} = V$ of N^* such that each N_{3i+1} equivariantly β_1 -deforms into N^* within N_{3i+2} by a deformation r_{it} rel N^* . Such a sequence exists because N and N^* are G -ANR's. Let β_2 be any refinement of β_1 such that the β_2 -neighborhood of each \overline{N}_i is contained in N_{i+1} , and let β be small enough that any β -equivalence f as stated is a β_2 -equivalence by maps and homotopies which extend the given homeomorphism $f|_{M^*}$ and its inverse (i.e., β is a star refinement of β_2). Write g for the homotopy inverse just described. Set $V_{1,0} = N_0$ and $V_{1,i} = N_{3i+3}$ for $i = 1, 2$.

Given f and U as above, and for any choice of $V' \subset V_{1,0}$, choose neighborhoods $Z_0 \subset \cdots \subset Z_3 = U \cap f^{-1}(V')$ of M^* such that for each $i < 3$ there is an ambient G - $f^{-1}\beta_2$ -deformation, D_{it} , of M rel M^* , supported on Z_{i+1} , which carries Z_i onto M^* (again by ANR-ness). Let $h_t: 1_M \sim gf$ be a β_2 -homotopy rel M^* . Then successive applications of D_{it} , h_t and $gr_{it}f$ to a map $j: (K, L) \rightarrow (f^{-1}N_{3i}, Z_i)$ provide a homotopy in $(f^{-1}N_{3(i+1)}, Z_{i+1})$ to a map into Z_{i+1} , with the homotopy limited by $f^{-1}(\text{St}^4(\beta_1))$. Thus, there is a $f^{-1}(\text{St}^8(\beta_1))$ -homotopy relative to L . Take $W_0 = Z_0$ and $W_i = Z_{i+1}$ for $i = 1, 2$.

As M^* is 1- LC in M , the second two stages satisfy the hypothesis for an engulfing structure, and we need only consider $h: (K^2, L) \rightarrow (f^{-1}V_{1,0} - M^*, W_0 - M^*)$. Let $h: K \times I \rightarrow F^{-1}N_3$ be the $f^{-1}(\text{St}^8\beta_1)$ -homotopy given above, and triangulate $K \times I$ cylindrically so that each simplex is contained in a linear chart refining $f^{-1}\beta$. By the fine 1-equivalence we may assume that the 2-skeleton of $K \times I$ misses M^* , and by coning in the chart neighborhoods we may assume that each 3-cell, σ , meets M^* in at most one (interior) point $\dot{\sigma}$.

Let $\Sigma = \bigcup_{\sigma} \dot{\sigma}$ and for simplicity assume that $\Sigma \subset K \times I \xrightarrow{\text{proj}} K$ is an embedding, and identify $\Sigma \times I$ with the union of the vertical segments from $\Sigma = \Sigma \times 0$ to $K \times 1$.

First adjust h so that each $\dot{\sigma} \times I$ intersects M^* only at σ and $\Sigma \times (0, 1]$ is locally polyhedrally embedded in $M - M^*$. Identifying I with the face v_0v_1 of the standard 2-simplex $\Delta^2 = (v_0, v_1, v_2)$ let

$$h': (\Sigma \times \Delta^2, \Sigma \times v_0v_2 \Sigma \times v_1v_2) \rightarrow (f^{-1}v_{1,0}; M^*, W_1)$$

be the extension of $h|_{\Sigma \times I}$ obtained from the $f^{-1}(\text{St}^8\beta_1)$ -homotopy above, and adjusted rel $\Sigma \times (v_0v_1 \cup v_0v_2)$ to embed $\Sigma \times (\Delta^2 - v_0v_2)$ locally polyhedrally in $M - M^*$. Now approximate $h: K \times I - \Sigma \rightarrow f^{-1}V_{1,0} - M^*$ rel $\Sigma \times (0, 1]$ so that $K \times I - \Sigma \times I$ is in general position with respect to $h'(\Sigma \times (\Delta^2 - v_0v_2))$ and the approximation extends continuously via $h|_{\Sigma}$.

By a move reminiscent of the "Alexander trick," we may slide the arcs $\dot{\sigma} \times I$ along the discs $\dot{\sigma} \times \Delta^2$, rel $\dot{\sigma}$, obtaining a new h whose intersection with M^* is precisely $\Sigma \times I$. Precomposing h with a slide in the domain which pulls $K \times I$ off a small regular neighborhood of $\Sigma \times I$, we obtain a homotopy as desired which misses M^* .

The choice of $V_{0,n-2} \subset \cdots \subset V_{0,0}$ is similar to Quinn's construction of an equivariant $(\eta, n-2, \emptyset)$ tame structure of the $N - N^*$ over N^* [Q₁]. (Quinn observes in [Q₂, 2.2.1] that such a structure exists. We simply modify his proof for our purposes, and refer the reader to [Q₁] for the necessary definitions and ingredients.) First choose $V_{0,0} \subset V_{1,0}$, a retraction $e: V_{0,0} \rightarrow N^*$, and an open G -cover η' of N^* so that $e^{-1}\eta'$ refines η . By the locally compact version of Quinn's eventual Hurewicz theorem [Q₁, 5.2], there is an open G -cover, μ , of N^* , such that the composite of a suitable sequence of "relatively homologically $(\mu, n-3)$ trivial" maps over N^* is "relatively $(\eta', n-3)$ connected." Notice that since Quinn's proof of the eventual Hurewicz theorem proceeds skeleton by skeleton it applies verbatim to equivariant maps of free G -pairs over N^* . We only give the construction of $V_{0,2} \subset V_{0,1} \subset V_{0,0}$, as the others are built analogously.

As in [Q₁, 5.3] it suffices to construct

$$V_{0,0} = V_{0,0,-2} \supset V_{0,1,-1} \supset V_{0,1,0} = V_{0,1} \supset V_{0,1,1} \supset \cdots \supset V_{0,1,2^{k+2}} = V_{0,2},$$

where $k = n-3$, such that

(1) each $f^{-1}(\overline{V}_{0,1,i} - \overline{V}_{0,1,i+1}) \subset f^{-1}(\overline{V}_{0,1,i-1} - \overline{V}_{0,1,i+2})$ is $(\mu, 1)$ -connected over N^* , and

(2) each $f^{-1}(\overline{V}_{0,1,i+1})$ is $(ef)^{-1}\mu$ -deformable into $M^* \text{ rel } M^*$, with the deformation taking place in $f^{-1}(V_{0,1,i})$ (giving rise to a controlled homological tame structure by the argument of [Q₁, 3.1.1]).

Quinn shows how to do this [Q₁, 5.4] when f is the identity map. We use Quinn's argument in N with a suitable refinement, μ' , of μ , together with buffer neighborhoods and β -homotopy data as in the argumentation for case (a) above to deduce that (1) and (2) hold in M .

4. Proof of the equivariant CE-mapping theorem. We first note that both here and in the proof of the equivariant α -approximation theorem we may assume, by passage to the double and Corollary 1.2, that our manifolds are without boundary. We make this assumption both here and in §5.

As shown in [Si₁], the complement follows from the main statement by Corollary 1.3, so it suffices to show approximability by homeomorphisms. We argue stratum by stratum, noting that since a G -CE map $f: M \rightarrow N$ restricts on fixed-point components $f_\gamma^H: M_\gamma^H \rightarrow N_\gamma^H$ to a W_γ -CE map (where W_γ is the isotropy subgroup of $\gamma \in \pi_0 M^H$ under the action of $W(H) = N(H)/H$), the dimensions of M_γ^H and N_γ^H are equal by the Vietoris-Begle theorem [Sp].

Our argument uses the inequivariant CE mapping theorem [Si₁], Lemma 3.1, and the equivariant Bing shrinking criterion. The statement and proof of the latter is a trivial generalization of the inequivariant version in Chapman [C₃, Theorem 26.1].

LEMMA 4.1 (EQUIVARIANT BING SHRINKING CRITERION). *Let $f: X \rightarrow Y$ be a proper surjective G -map between locally compact G -spaces. Then f is an equivariant near homeomorphism if and only if for each open G -cover \mathcal{U} of X and each open G -cover \mathcal{V} of Y there is an equivariant homeomorphism $h: X \rightarrow X$ such that fh is \mathcal{V} close to f and each $h(f^{-1}(y))$ lies in some element of \mathcal{U} .*

Thus, assume, after taking a suitable approximation, that f is a homeomorphism on all strata smaller than M_γ^H . We shall approximate f by a map which is a homeomorphism $M_\gamma^H \rightarrow N_\gamma^H$.

Let \mathcal{D} be the decomposition of M whose only nontrivial elements are the sets $M_\gamma^H \cap f^{-1}y$ for $y \in N_\gamma^{>H}$ and their G -translates. Then f factors through a G -CE map $\bar{f}: M/\mathcal{D} \rightarrow N$. We shall show that the projection map $\pi: M \rightarrow M/\mathcal{D}$ is a near homeomorphism, so that we may assume in the subsequent argument that $f^{-1}N_\gamma^{>H} \cap M_\gamma^H = M_\gamma^{>H}$.

Let \mathcal{U} and \mathcal{V} be covers of M and M/\mathcal{D} as in the hypothesis of Lemma 4.1. Let \mathcal{V}' be an equivariant collection of open sets in N which covers $GN_\gamma^{>H}$, such that $\bar{f}^{-1}\mathcal{V}'$ refines \mathcal{V} and let V be a G -neighborhood of $GN_\gamma^{>H}$ in N whose closure is covered by \mathcal{V}' .

Let U be a G -neighborhood of $GM_\gamma^{>H}$ in M such that $U \cap f^{-1}\mathcal{V}'$ refines \mathcal{U} . The main lemma provides a W_γ -equivariant $f^{-1}\mathcal{V}'$ -isotopy, h_t , of $M_\gamma^H \text{ rel } M_\gamma^{>H}$, supported on $f^{-1}V$, such that each set in $\mathcal{D} \cap M_\gamma^H$ is contained in $h_1(U \cap M_\gamma^H)$. As h_t is constructed from small pushes in $M_\gamma^H - M_\gamma^{>H}$, Corollary 1.7 provides an extension of h to an equivariant $f^{-1}\mathcal{V}'$ -isotopy of 1_M supported on $f^{-1}V$ (hence a $\pi^{-1}\mathcal{V}$ -isotopy) \bar{h}_t . The required shrinking is given by \bar{h}_1^{-1} .

Thus, we shall assume that $f^{-1}N_\gamma^{>H} \cap M_\gamma^H = M_\gamma^{>H}$. Let \mathcal{D}' be the decomposition of M whose only nontrivial elements are $M_\gamma^H \cap f^{-1}y$ for $y \in N_\gamma^H - N_\gamma^{>H}$ and their G -translates. As above, it suffices to show that $\pi: M \rightarrow M/\mathcal{D}'$ is a near homeomorphism.

Given \mathcal{U} and \mathcal{V} as above, choose refinements \mathcal{U}' and \mathcal{V}' of $\mathcal{U} \cap f^{-1}(N - GN_\gamma^{>H})$ and $\mathcal{V} \cap \bar{f}^{-1}(N - GN_\gamma^{>H})$, respectively, with \mathcal{V}' sufficiently fine that any self-map of $M - f^{-1}GN_\gamma^{>H}$ which is $\pi^{-1}\mathcal{V}'$ -close to the identity map extends continuously by the identity map of $f^{-1}GN_\gamma^{>H}$.

Consider $\pi: M_\gamma^H - M_\gamma^{>H} \rightarrow (M_\gamma^H - M_\gamma^{>H})/\mathcal{D} \cong N_\gamma^H - N_\gamma^{>H}$. Here, W_γ acts freely and the induced map of orbit spaces is CE. Lifting the homotopy obtained from the inequivariant complement of the CE mapping theorem, we obtain a W_γ -equivariant homotopy f_t , with $f_1 = f$ and f_t a homeomorphism for $0 \leq t < 1$, with the homotopy limited by $f^{-1}\mathcal{V}''$, for \mathcal{V}'' a star refinement of \mathcal{V}' . Then for any $t_0 < 1$, $f_0^{-1}f_t$ is a $\pi^{-1}\mathcal{V}'$ -isotopy from the identity map to $f_0^{-1}f_{t_0}$. For t_0 sufficiently close to 1, $f_0^{-1}f_{t_0}$ shrinks sets in \mathcal{D}_1 into sets in \mathcal{U}' . Moreover, the construction of $f_0^{-1}f_t$ given by [Si₁] is as a composite of compactly supported moves, so an equivariant extension $g: M \rightarrow M$ of $f_0^{-1}f_{t_0}$ exists by Corollary 1.7, and shrinks \mathcal{D}_1 as required.

5. Proof of the equivariant α -approximation theorem. As above, we may assume that our manifolds are without boundary, and the following lemma gives the dimension requirements for a stratum by stratum induction.

LEMMA 5.1. *Let α be an open G -cover of the G -manifold N^n . Then there is an open G -cover β of N such that if $f: M^n \rightarrow N^n$ is a G - β -equivalence which restricts to a homeomorphism on all strata of dimension less than 5, then the restriction to each fixed-point component $f_\gamma^H: M_\gamma^H \rightarrow N_\gamma^H$ is α -close to an inequivariant homeomorphism.*

PROOF. Since f_γ^H is a W_γ - β -equivalence, [C₁, Theorem 1] provides that for β sufficiently small, f_γ^H is close to an inequivariant approximate fibration. Since the

homotopy fiber of f_γ^H is trivial, this approximate fibration is a cell-like map. The result follows from [Si₁] and the Vietoris-Begle theorem [Sp].

In fact, this sketch is a paradigm for our argument. We show that if $f_\gamma^{>H}$ is already a homeomorphism, α is given, and β is sufficiently small, then f_γ^H is α -close to a W_γ -CE map. Our constructions extend equivariantly over M with control, and the induction is completed by the (inductive) proof of the G -CE mapping theorem.

For simplicity, we replace $M_\gamma^H, N_\gamma^H, M_\gamma^{>H}, N_\gamma^{>H}$ and W_γ by M, N, M^*, N^* and G , respectively.

As mentioned in the introduction, we shall adapt Chapman's argument [C₁] to show that for any α there is a β such that if $f: M \rightarrow N$ is a G - β -1 equivalence, with $f^*: M^* \rightarrow N^*$ a homeomorphism, then for any open G -cover γ of N there is a G - γ -equivalence $f': M \rightarrow N$ which is α -close to f . The required G -CE map is then obtained as a limit of a convergent sequence of such f' .

REMARKS. Our engulfing techniques permit a full equivariant analogue of Chapman's Theorem 1 (approximation by approximate fibrations) if M has "stable" gaps (i.e., $2 \dim M_\gamma^{>H} < \dim M_\gamma^H$ whenever $M_\gamma^{>H} \neq M_\gamma^H$), by using topological dual skeleta of dimension less than the gap codimension and general position.

Our adaptation of Chapman's handle lemma [C₁, Theorem 5.2] is the following. Here, generalized singular set corresponds to M^*, N^* , etc.

LEMMA 5.2. *For each $\varepsilon > 0$ and $n \geq 0$ there is a $\delta > 0$ such that for each $\mu > 0$ there exists $\nu > 0$ such that if M^n is a G -manifold and $f: M \rightarrow \mathbf{R}^{n-k} \times \mathbf{R}_\rho^k$ is a homeomorphism on generalized singular sets, an equivariant δ -equivalence over $B_3^{n-k} \times D_3^k$ and an equivariant ν equivalence over $B_3^{n-k} \times (D_3^k - \dot{D}_{1/3}^k)$, then there is a map $\tilde{f}: M \rightarrow \mathbf{R}^{n-k} \times \mathbf{R}_\rho^k$ which is an equivariant μ -fibration over $B_1^{n-k} \times D_1^k$ and is equivariantly ε -homotopic to $f \text{ rel } M - f^{-1}(\dot{B}_3^{n-k} \times \dot{D}_{2/3}^k)$.*

Here B_t^{n-k} is the unit ball of radius t in \mathbf{R}^{n-k} and D_t^k is the unit ball of radius t in \mathbf{R}_ρ^k .

The argument now follows by a standard induction down the dual handle structure in each element of a cover of N by charts. For a cell σ^{n-k} of type H , we apply the handle lemma H -equivariantly to $\mathbf{R}^{n-k} \times \mathbf{R}_\rho^k$, where \mathbf{R}_ρ^k is the open dual cell of σ and \mathbf{R}^{n-k} lies in the interior of σ , and obtain G -equivariance by translation. Thus, the induction starts with the action of the trivial group on \mathbf{R}^n . Here, there is no statement about ν , but a direct application of Chapman's Lemma 5.1 produces a map which is a μ -fibration over B_1^n . In fact, by our codimension 3 gap hypothesis, equivariance does not enter until $k = 3$. At each stage, μ is the ν produced by the previous stage and the β -control has been relaxed by a factor of 2ε . For a given ε and small enough β , we get a small deformation to a μ -fibration for any μ as required.

The handle lemma is proven by Chapman's shuffle argument, in which the problem is first localized over an embedded $T^{n-k-1} \times \mathbf{R} \times \mathbf{R}_\rho^k \subset \mathbf{R}^{n-k} \times \mathbf{R}_\rho^k$, wrapped up over $T^{n-k} \times \mathbf{R}_\rho^k$, and unfurled over $\mathbf{R}^{n-k} \times \mathbf{R}_\rho^k$. Chapman uses the unfurled map and a radial contraction of the base to get improved control over part of the target, and controlled engulfing to cover a translation shifting the product of unit balls into the region of improved control.

The controlled engulfing for both this move and the wrapping up is accomplished by the following lemma. Here we are given $D = \mathring{D}_\rho^{n-1}$, and equivariant map $\phi: D \rightarrow [0, \infty)$, and an isotopy $\theta_t: \mathbf{R} \rightarrow \mathbf{R}$, $t \geq 0$, supported on $[-1, 1]$, with $\theta_t = \text{id}$ for $t \geq 1$. Let $\theta: D \times \mathbf{R} \rightarrow D \times \mathbf{R}$ be given by $\theta(x, y) = (x, \theta_{\phi(x)}(y))$, let $D_t = \phi^{-1}([0, t])$ and let $\pi: D \times \mathbf{R} \rightarrow D$ by the projection.

LEMMA 5.3. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $f: M^n \rightarrow D \times \mathbf{R}$ is an equivariant map which is a G - δ -equivalence over $D_4 \times [-4, 4]$ and a homeomorphism on generalized singular sets, then there is an equivariant $(\pi f)^{-1}$ ε -isotopy $\tilde{\theta}_t$ of the identity map of M , supported on $f^{-1}(D_2 \times [-2, 2])$ such that $f\tilde{\theta}_1$ is ε -close to θf . Additionally, if $\mu > 0$ there is a $\nu > 0$ such that if f is a ν -equivalence over $(D_4 - \mathring{D}_{1/3}) \times [-4, 4]$, then we may require that $f\tilde{\theta}_1$ is μ -close to θf over $(D_4 - \mathring{D}_{2/3}) \times \mathbf{R}$.*

PROOF. In [C₁, Theorem 3.6] Chapman works his analogue of this up from an easier lemma by the method of partitioning $[-1, 1]$ as $-1 = x_0 < \dots < x_k = 1$ and considering the laminae $\theta(D \times [x_{i-1}, x_i])$ and the tubes $\pi^{-1}\beta$ for a sufficiently fine open cover β of D . He obtains his homeomorphism $\tilde{\theta}_1$ by using engulfing to move each $f^{-1}(D \times x_i)$ into $f^{-1}\theta(D \times (x_{i-1}, x_i))$ with movement guided by the tubes $(\pi f)^{-1}\beta$. The result is, for sufficiently fine partitions and sufficiently fine β , sufficiently well controlled. (This is because the control is mediated by the thickness of the laminae and by $\text{St}^p\beta$, where $p = p(n, k)$ comes from repeated engulfing control estimates. Thus, given μ , we can choose a sufficiently fine partition, and choose β to be sufficiently fine near $D - \mathring{D}_{2/3}$, that the laminar chambers of $\pi^{-1}(\text{St}^p\beta)$ in $(D - \mathring{D}_{2/3}) \times \mathbf{R}$ have diameter less than $\frac{1}{3}\mu$. If we now choose ν so small that each subset, A , of $(D - \mathring{D}_{2/3}) \times [-1, 1]$ of diameter $< \nu$ is contained in $\beta_i \times [x_{i-1}, x_i]$ for some $\beta_i \in \beta$, then the diameter of $\theta(A) < \mu$ and the ν -equivalence stays a μ -equivalence.)

All of the above considerations go over verbatim to the equivariant context once we establish the equivariant analogue of Chapman's simpler lemma [C₁, Lemma 3.4], which is as follows, using the notation of Lemma 5.3 above.

LEMMA 5.4. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $f: M^n \rightarrow D \times \mathbf{R}$ is an equivariant map which is a G - δ -equivalence over $D \times [-4, 4]$ and a homeomorphism on generalized singular sets, then there is an equivariant $(\pi f)^{-1}$ ε -isotopy, h_t , of the identity map of M , supported on $f^{-1}(D \times [-3, 3])$ such that*

$$f^{-1}(D \times (-\infty, 1]) \subset h_1 f^{-1}(D \times (-\infty, 0)).$$

PROOF. Let $\gamma > 0$ be given. By induction up the strata of D^* , applying Corollary 1.7 to isotopies obtained by feathering out (by partitions of unity) piecewise linear slides in the \mathbf{R} -coordinate of $D^* \times \mathbf{R} \cong M^*$, we may assume given an equivariant $(\pi f)^{-1}$ γ -isotopy θ_t of the identity map of M , supported on $f^{-1}(D \times [-3, 2])$ such that on some neighborhood, U , of M^* ,

$$\theta_1(U \cap f^{-1}(D \times [-2, \infty))) \subset f^{-1}(D \times (\frac{3}{2}, \infty)).$$

By the proof of the main lemma, which works perfectly well “over $D \times [-4, 4]$,” there is a G -neighborhood, V , of $D^* \times [-\frac{5}{2}, \frac{5}{2}]$ and a $\beta > 0$ such that if f is a β -equivalence over $D \times [-4, 4]$ and U is as above, there is an equivariant $f^{-1}\gamma$ -isotopy, λ_t , of the identity map of M , supported on $f^{-1}(D \times [-3, 3])$, such that $\lambda_1(f^{-1}V) \subset U$.

If f is a δ -equivalence over $D \times [-4, 4]$ and μ_t is any homotopy of $1_{D \times \mathbf{R}}$ supported on $D \times [-4, 4]$, then an equivariant δ -lift $\tilde{\mu}_t: M \rightarrow M$ will exist and will be isovariant off f^{-1} of the δ -neighborhood of $D^* \times \mathbf{R}$. We can use this observation together with our controlled engulfing (cf. [C₁, Lemma 3.2]) to produce a $(\pi f)^{-1} \xi$ -isotopy, $\tilde{\mu}_t$, of the identity map of M , supported on $f^{-1}(D \times [-\frac{3}{2}, 3] - V_1)$, for some neighborhood $V_1 \subset V$ of $D^* \times [-\frac{3}{2}, \frac{5}{2}]$ such that $\tilde{\mu}_1 f^{-1}(D \times [-1, \infty)) \cap f^{-1}(D \times (-\infty, \frac{9}{4}])$ is contained in $f^{-1}V$. (Here, ξ depends on δ via Corollary 2.3.)

Since λ_t has control over $D \times \mathbf{R}$, a small choice of γ will guarantee that

$$\lambda_1 f^{-1}(D \times [\frac{9}{4}, \infty))$$

is in the complement of the support of θ_t and that $\lambda_1 \tilde{\mu}_1(f^{-1}(D \times [-1, \infty))) \subset f^{-1}D \times [-2, \infty)$. Thus if γ, ξ and $\delta < \beta$ are small, successive applications of $\tilde{\mu}_t, \lambda_t$ and θ_t provide the desired $(\pi f)^{-1} \varepsilon$ -isotopy.

6. β -dominations and β -maps. Ferry proves the inequivariant versions of the β -domination theorem and Corollary 2 as a straightforward application of the following two results. We shall state them and discuss the required modifications of Ferry’s proofs.

LEMMA 6.1. *Let $Y \subset X$ be a locally compact G -ANR pair with Y closed in X . Then for any open G -cover α of X there is an open G -cover β of X such that if $g: X \rightarrow Z$ is a surjective proper G - β -map, then there is a proper α -domination of pairs $f: (Z, g(Y)) \rightarrow (X, Y)$ with right α -inverse g .*

LEMMA 6.2. *Let α be an open G -cover of the G -manifold M^n . Then there is an open G -cover β of M such that if N^n is a G -manifold and $f: (N, \partial N) \rightarrow (M, \partial M)$ is a proper G - β -domination of pairs for which $\dim N_\gamma^H = \dim M_{f(\gamma)}^H$ for all fixed-point components, N_γ^H , of N , then f is a G - α -equivalence of pairs.*

Ferry proves Lemma 6.1 using the convex structure of separable Hilbert space, l_2 , and the fact that any locally compact (separable metric) ANR embeds as a closed subspace of l_2 . The equivariant analogue is $l_2[G]$, the separable Hilbert space whose basis is a free G -set. Embeddability of locally compact G -ANR’s in $l_2[G]$ follows from the usual arguments together with the embedding theory in Bredon [Br] (cf. [SW₆]).

The first requirement for generalizing Ferry’s proof of Lemma 6.2 is that any n -dimensional G -ANR be equivariantly β -dominated by an n -dimensional G -CW complex. This follows from the usual nerves-of-covers arguments. The only change necessary is that the application of Ferry’s eventual Hurewicz theorem [F, Propositions 3.1–3.3] requires a sequence of neighborhoods for each orbit type in the local representaton D_ρ , as it will not necessarily be possible to select a single sequence of G -connected neighborhoods. Thus, we argue by induction on orbit types and consider only the restriction of the neighborhoods in question to the given fixed-point set.

7. The equivariant bundle theorem. Let $p: E \rightarrow B$ be a proper equivariant surjection of locally compact, separable metric G -spaces. We say that p is equivariantly completely regular if for each $\varepsilon > 0$ and each $x \in B$ there is a slice neighborhood U of x such that for each $y \in U$ there is a G_y -homeomorphism $p^{-1}(y) \rightarrow p^{-1}(x)$ which is ε -close to the inclusion $p^{-1}(y) \subset E$.

We show that a G -fibration satisfying the hypothesis of Theorem 2 is equivariantly completely regular, and deduce Theorem 2 from Corollary 1.3 and the following generalization of a theorem of Dyer and Hamstrom [DH], which is based on Michael's selection theorem [M].

PROPOSITION 7.1. *Let $p: E \rightarrow B$ be equivariantly completely regular, with B locally finite dimensional, and suppose that the homeomorphism group of each $p^{-1}(x)$ is locally G_x -contractible. Then p is a G -bundle.*

PROOF. We wish to find an equivariant local trivialization with the diagonal action over a slice around $x \in B$, so we may as well restrict attention to a slice over which each $p^{-1}y$ is G_y -homeomorphic to $p^{-1}x$. For simplicity, call this slice B , write G for G_x and write F for $p^{-1}x$.

Let $\mathcal{P}E$ be the subspace of the space of maps from F to E which consists of homeomorphisms from F to fibers of p and let $\pi: \mathcal{P}E \rightarrow B$ send $h: F \rightarrow p^{-1}y$ to y . Let G act on $\mathcal{P}E$ by the diagonal action: $(g \cdot h)(z) = g(h(g^{-1}z))$. Then π is equivariant, and if $s: U \rightarrow \mathcal{P}E$ is an equivariant section of π over a neighborhood U of x in B , then there is an equivariant local trivialization (with diagonal action) $\hat{s}: U \times F \rightarrow p^{-1}U$ given by $\hat{s}(y, z) = s(y)(z)$. Conversely, an equivariant local trivialization induces a local section.

Let $\mathcal{P}'E \subset \mathcal{P}E$ be the subspace consisting of those $h: F \rightarrow p^{-1}y$ which are G_y -equivariant. Then the image of an equivariant local section of π always lies in $\mathcal{P}'E$. In fact, $\mathcal{P}'E \cap \pi^{-1}y$ is the G_y -fixed point set of $\pi^{-1}y$, and the restriction $\pi': \mathcal{P}'E \rightarrow B$ of π is isovariant. Thus, the equivariant local sections of π' are in 1-1 correspondence to the local sections of the orbit map $\pi'/G: \mathcal{P}'E/G \rightarrow B/G$.

We wish to apply Michael's selection theorem [M, Theorem 1.2]. In his language, a selection for the carrier $\phi(y) = (\pi'/G)^{-1}(y)$ is precisely a section of π'/G . Local sections will exist provided that ϕ is lower semicontinuous and that the family of fibers of π'/G is equi- LC^n for $n > \dim B$.

The former condition is equivalent to showing that π'/G is an open map. Since projections to orbit spaces are open, it suffices to show that π' is open. Let V be an open subspace of $\mathcal{P}'E$ and let $\{y_i\}$ be a sequence in $B - \pi'(V)$ which converges to $y \in B$. By equivariant complete regularity, after possibly passing to a subsequence of $\{y_i\}$, there is a sequence of homeomorphisms $f_i: p^{-1}y \rightarrow p^{-1}y_i$, such that f_i is G_{y_i} -equivariant, which converges to the identity map of $p^{-1}y$ (recall that p is proper). If $y \in \pi'(V)$, there is a G_y -homeomorphism $f: F \rightarrow p^{-1}y$ in V . But then $\{f_i f\}$ converges to f in $\mathcal{P}'E$.

It now suffices to show that the fibers of π' are equivariantly equi- LC^n in $\mathcal{P}'E$, i.e., for each G_y -neighborhood U of $f \in \pi'^{-1}y$ in $\mathcal{P}'E$ there is a G_y -neighborhood $V \subset U$ such that any map of a k -sphere into $V \cap \pi'^{-1}(z)$, $k \leq n$, is nullhomotopic in $U \cap \pi'^{-1}(z)$ for any $z \in B$.

Let \tilde{V} be a G_y -neighborhood of f in $\mathcal{P}E$ such that $\tilde{V} \cap \mathcal{P}'E = V$ and let V' be a G_y -neighborhood of f in $\pi^{-1}y$ such that the ε -neighborhood of V' is contained

in \tilde{V} for some $\varepsilon > 0$. Let U' be a neighborhood of f in $\pi^{-1}y$ which G_y -contracts in V' and suppose that the 2δ -neighborhood of f in $\pi^{-1}y$ is contained in U' for $\delta < \varepsilon$. Take U to be the intersection of the δ -ball around f with the inverse image under π' of the neighborhood around y specified for δ in the definition of equivariant complete regularity. Since left translation by G_y -homeomorphisms is G_y -equivariant in $\mathcal{P}E$, the result follows.

We now show that a G -fibration as stated is equivariantly completely regular. This follows from the equivariant α -approximation theorem, provided that p satisfies the obvious equivariant analogue of a strongly regular map (cf. [CF]), which holds provided that over a compact neighborhood of each $x \in B$ there is an equivariant lifting function such that for each $\varepsilon > 0$ there is a $\delta > 0$ such that equivariant δ -homotopies into B lift to equivariant ε -homotopies into E . Since p is proper, this will hold by a Lebesgue number argument for any equivariant regular lifting function. Since B is metric, such a lifting function exists by the usual arguments (cf. [SW₁]).

REMARKS. In fact, Michael's argument has an obvious generalization to actions by finite groups, which would provide equivariant sections of π directly.

8. The equivariant thin h -cobordism theorem. Let (W, M) be a G - βh -cobordism with controlling deformation retraction $r: W \rightarrow M$. Let \bar{W} be the union of W with an external collar on M and let $\bar{r}: \bar{W} \rightarrow M \times I$ be r on W and the identity map on the collar. Since r is close to the projection map on the low dimensional strata, we may assume, with slight loss of control, that \bar{r} is a homeomorphism there. Thus, our thin h -cobordism will follow from the α -approximation theorem provided that \bar{r} is a well enough controlled homotopy equivalence on the boundary.

It is enough to show that for β small, the inclusion $\overline{\partial W - M} \subset W$ is a G - $r^{-1}\gamma$ -equivalence for γ small. First, replace $\overline{\partial W - M}$ with M_1 , the complement in ∂W of the union of M with a small (with respect to β) open collar on ∂M .

We shall apply the locally compact version of Quinn's eventual Hurewicz theorem. As before, since Quinn's argument is an induction on skeleta, it suffices to show that Quinn's hypotheses hold simultaneously in each fixed-point set. For this we note that since W is locally linear and local representations are constant along isovariant arcs, a quick induction shows that each fixed-point component of W is an equivariant β - h -cobordism between the pertinent components of M and M_1 .

Consider the end $r: W - M \rightarrow M$ and let $U_0 \supset \cdots \supset U_{2n+3}$ be complements of small collar neighborhoods of M_1 in W . Then for β' slightly larger than β there is a G - β' -deformation retraction of each U_i onto M . Moreover, each inclusion $U_j - U_{j+1} \subset U_{j-1} - U_{j+2}$ is equivariantly $(\beta', 1)$ -connected because the identity map of $\overline{\partial W - M}$ is $(\beta, 1)$ -connected. Thus, as in [Q₁, 5.3], there is a small equivariant deformation of $W - M$ rel M_1 into a small collar around M_1 .

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